Interplay between Zamolodchikov–Faddeev and reflection–transmission algebras

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Abstract
We show that a suitable coset algebra, constructed in terms of an extension of the Zamolodchikov–Faddeev algebra, is homomorphic to the reflection–transmission algebra, as it appears in the study of integrable systems with impurity.

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1. Introduction

The Zamolodchikov–Faddeev (ZF) algebra [1, 2] is considered to be the basis of factorized scattering theory in integrable models. The two-body scattering, which is the only particle interaction present in this case, is implemented by a quadratic constraint among the particle creation and annihilation operators. There exist various attempts [3–10] to generalize factorized scattering theory to the case when point-like impurities, preserving integrability, are present as well. In this context, the relevant algebraic structure emerging recently [11, 12], is the so-called reflection–transmission (RT) algebra. Besides the two-body scattering in the bulk, the RT algebra also captures the particle interaction with the impurity. From the physical analysis performed in [11], it follows that the particle–impurity interactions are implemented by further constraints, ensuring compatibility between bulk scattering and the process of reflection/transmission from the impurity. This feature suggests the existence of an algebraic connection among ZF and RT algebras. It turns out in fact that taking the coset of the ZF algebra with respect to a suitable two-sided ideal, one obtains an algebra homomorphic to the RT algebra. The explicit realization of this relationship is the main goal of this investigation. Our analysis extends the results of [13, 14], obtained in the case of pure reflection. Establishing the link between ZF and RT algebras, we take the occasion to discuss also the issue of symmetries when impurities are present.

2. Background

We collect here some definitions and basic results concerning ZF and RT algebras, which are needed in what follows. Adopting throughout the paper the compact tensor notation, introduced in [15], we start with the following definition.

Definition 2.1 (ZF algebra $\mathcal{A}_S$). *$\mathcal{A}_S$ is the polynomial algebra generated by a unit element $\mathbf{1}$ and the generators $a(k)$ and $a^\dagger(k)$, subject to the constraints*

$$a_1(k_1)a_2^\dagger(k_2) = a_2^\dagger(k_2)S_{12}(k_1,k_2)a_1(k_1) + \delta_{12}\delta(k_1 - k_2)\mathbf{1} \tag{2.1}$$

$$a_1(k_1)a_2(k_2) = S_{21}(k_2,k_1)a_2(k_2)a_1(k_1) \tag{2.2}$$

$$a_1^\dagger(k_1)a_2^\dagger(k_2) = a_2^\dagger(k_2)a_1^\dagger(k_1)S_{21}(k_2,k_1). \tag{2.3}$$

Here and below the S-matrix obeys the following well-known relations (Yang–Baxter equation and unitarity):

$$S_{12}(k_1,k_2)S_{13}(k_1,k_3)S_{23}(k_2,k_3) = S_{23}(k_2,k_3)S_{13}(k_1,k_3)S_{12}(k_1,k_2) \tag{2.4}$$

$$S_{12}(k_1,k_2)S_{21}(k_2,k_1) = \mathbb{I} \otimes \mathbb{I}. \tag{2.5}$$

We emphasize that S depends in general on χ_1 and χ_2, observing in addition that bulk S-matrices depending on $\chi_1 - \chi_2$ only are also covered by our framework, which therefore allows us to treat both systems with exact as well as broken Lorentz (Galilean) invariance. More details about the physical meaning of this generalization are given in [12].

We shall employ below the extension $\overline{\mathcal{A}}_S$ of $\mathcal{A}_S$, involving power series in $a(k)$ and $a^\dagger(k)$ whose individual terms preserve the particle number. The concept of extended ZF (EZF) algebra $\overline{\mathcal{A}}_S$ is relevant for proving [15] the following.

Proposition 2.2 (Well-bred operators). *Each $\overline{\mathcal{A}}_S$ contains an invertible element $L(k)$ satisfying*

$$L_1(k_1)a_2(k_2) = S_{21}(k_2,k_1)a_2(k_2)L_1(k_1) \tag{2.6}$$

$$L_1(k_1)a_2^\dagger(k_2) = a_2^\dagger(k_2)S_{12}(k_1,k_2)L_1(k_1). \tag{2.7}$$

Moreover, $L(k)$ obeys

$$S_{12}(k_1,k_2)L_1(k_1)L_2(k_2) = L_2(k_2)L_1(k_1)S_{12}(k_1,k_2) \tag{2.8}$$

and generates a quantum group $\mathcal{U}_S \subset \overline{\mathcal{A}}_S$, with Hopf structure $\Delta L(k) = L(k) \otimes L(k)$. $L(k)$ also satisfies $L(k)^\dagger = L(k)^{-1}$.

$L(k)$, called a well-bred operator, is explicitly constructed in [15] and admits a series representation in terms of $a(k)$ and $a^\dagger(k)$.

We turn now to RT algebras [12].

Definition 2.3 (RT algebra $\mathcal{C}_S$). *A RT algebra is generated by $\mathbf{1}$ and the generators $A(k)$, $A^\dagger(k)$, $t(k)$ and $r(k)$ obeying*

$$A_1(k_1)A_2(k_2) = S_{21}(k_2,k_1)A_2(k_2)A_1(k_1) \tag{2.9}$$

$$A_1^\dagger(k_1)A_2^\dagger(k_2) = A_2^\dagger(k_2)A_1^\dagger(k_1)S_{21}(k_2,k_1) \tag{2.10}$$

$$A_1(k_1)A_2^\dagger(k_2) = A_2^\dagger(k_2)S_{12}(k_1, k_2)A_1(k_1) + \delta(k_1 - k_2)(\delta_{12}\mathbf{1} + t_{12}(k_1)) + r_{12}(k_1)\delta(k_1 + k_2)$$

$$(2.11)$$

$$A_1(k_1)t_2(k_2) = S_{21}(k_2, k_1)t_2(k_2)S_{12}(k_1, k_2)A_1(k_1) \tag{2.12}$$

$$A_1(k_1)r_2(k_2) = S_{21}(k_2, k_1)r_2(k_2)S_{12}(k_1, -k_2)A_1(k_1) \tag{2.13}$$

$$t_1(k_1)A_2^\dagger(k_2) = A_2^\dagger(k_2)S_{21}(k_2, k_1)t_1(k_1)S_{21}(k_2, k_1) \tag{2.14}$$

$$r_1(k_1)A_2^\dagger(k_2) = A_2^\dagger(k_2)S_{21}(k_2, k_1)r_1(k_1)S_{21}(k_2, -k_1) \tag{2.15}$$

$$S_{12}(k_1, k_2)t_1(k_1)S_{21}(k_2, k_1)t_2(k_2) = t_2(k_2)S_{12}(k_1, k_2)t_1(k_1)S_{21}(k_2, k_1) \tag{2.16}$$

$$S_{12}(k_1, k_2)t_1(k_1)S_{21}(k_2, k_1)r_2(k_2) = r_2(k_2)S_{12}(k_1, -k_2)t_1(k_1)S_{21}(-k_2, k_1) \tag{2.17}$$

$$S_{12}(k_1, k_2)r_1(k_1)S_{21}(k_2, -k_1)r_2(k_2) = r_2(k_2)S_{12}(k_1, -k_2)r_1(k_1)S_{21}(-k_2, -k_1) \tag{2.18}$$

where $r(k)$ and $t(k)$ satisfy

$$t(k)t(k) + r(k)r(-k) = \mathbf{1} \tag{2.19}$$

$$t(k)r(k) + r(k)t(-k) = 0. \tag{2.20}$$

We refer to $r(k)$ and $t(k)$ as reflection and transmission generators.

A special case of RT algebra is the boundary algebra $\mathcal{B}_S$ [16] defined as the coset of $\mathcal{C}_S$ by the relation $t(k) = 0$. More precisely, one defines the left ideal $\mathcal{I}_S = \mathcal{C}_S \cdot \{t(k)\}$. From the above relations, this left ideal is obviously a two-sided ideal, so that the coset $\mathcal{C}_S/\mathcal{I}_S$ is an algebra: it is the boundary algebra $\mathcal{B}_S$, whose defining relations are given by equations (2.9)–(2.19) with $t(k) = 0$.

Note that the same construction applies when $r(k) = 0$, leading to an algebra associated with a purely transmitting impurity [17].

Note also that the generators $r(k)$ and $t(k)$ form a subalgebra $\mathcal{K}_S \subset \mathcal{C}_S$. This subalgebra itself contains two subalgebras. The subalgebra $\mathcal{R}_S \subset \mathcal{K}_S$, generated by $r(k)$, has already appeared in [18] and is called the reflection algebra. In the same way, $t(k)$ generates a subalgebra $\mathcal{T}_S \subset \mathcal{K}_S$, called in analogy the transmission algebra.

3. From $\overline{\mathcal{A}}_S$ to $\mathcal{C}_S$

Let $R(k)$ and $T(k)$ be any two matrix functions satisfying

$$R^\dagger(k) = R(-k) \qquad T^\dagger(k) = T(k) \tag{3.1}$$

$$T(k)T(k) + R(k)R(-k) = \mathbb{I} \tag{3.2}$$

$$T(k)R(k) + R(k)T(-k) = 0 \tag{3.3}$$

and the boundary Yang–Baxter equation

$$S_{12}(k_1, k_2)R_1(k_1)S_{21}(k_2, -k_1)R_2(k_2) = R_2(k_2)S_{12}(k_1, -k_2)R_1(k_1)S_{21}(-k_2, -k_1). \tag{3.4}$$

It has been shown in [12] that $R(k)$ and $T(k)$ obey in addition the *transmission* and *reflection–transmission* Yang–Baxter equations

$$S_{12}(k_1, k_2)T_1(k_1)S_{21}(k_2, k_1)T_2(k_2) = T_2(k_2)S_{12}(k_1, k_2)T_1(k_1)S_{21}(k_2, k_1) \tag{3.5}$$

$$S_{12}(k_1, k_2)T_1(k_1)S_{21}(k_2, k_1)R_2(k_2) = R_2(k_2)S_{12}(k_1, -k_2)T_1(k_1)S_{21}(-k_2, k_1). \tag{3.6}$$

This remarkable property of the pair $\{R(k), T(k)\}$ was discovered in [12] and is fundamental in what follows. In fact, by a direct calculation, starting from the EZF algebra $\overline{\mathcal{A}}_S$, one proves:

Proposition 3.1. *Let*

$$t(k) = L(k)T(k)L^{-1}(k) \tag{3.7}$$

$$r(k) = L(k)R(k)L^{-1}(-k) \tag{3.8}$$

where $L(k)$ is the well-bred operator of $\overline{\mathcal{A}}_S$ and $R(k)$ and $T(k)$ are defined above. Then, $a(k), a^\dagger(k), t(k)$ and $r(k)$ obey the relations (2.12)–(2.20).

Proposition 3.2. *The map*

$$\rho \begin{cases} a(k) \rightarrow \alpha(k) = t(k)a(k) + r(k)a(-k) \\ a^\dagger(k) \rightarrow \alpha^\dagger(k) = a^\dagger(k)t(k) + a^\dagger(-k)r(-k) \end{cases} \tag{3.9}$$

extends to a homomorphism on $\overline{\mathcal{A}}_S$.

Proof. Direct calculation using the commutation relations of the ZF algebra and the equations obeyed by $R(k)$ and $T(k)$. $\qquad\square$

Let us remark that we have the identities

$$a(k) = t(k)\alpha(k) + r(k)\alpha(-k) \tag{3.10}$$

$$a^\dagger(k) = \alpha^\dagger(k)t(k) + \alpha^\dagger(-k)r(-k). \tag{3.11}$$

However, since $t(k)$ and $r(k)$ are still expressed in terms of $a(k)$ and $a^\dagger(k)$, these are not really 'inversion formulae'.

Note that $\alpha(k), \alpha^\dagger(k), t(k)$ and $r(k)$ also obey the relations (2.12)–(2.20).

The homomorphism ρ is essential for constructing an RT algebra from EZF. Indeed, we have:

Theorem 3.3. *Let*

$$A(k) = \tfrac{1}{2}(a(k) + \alpha(k)) = \tfrac{1}{2}([1 + t(k)]a(k) + r(k)a(-k)) \tag{3.12}$$

$$A^\dagger(k) = \tfrac{1}{2}(a^\dagger(k) + \alpha^\dagger(k)) = \tfrac{1}{2}(a^\dagger(k)[1 + t(k)] + a^\dagger(-k)r(-k)). \tag{3.13}$$

Then $A(k), A^\dagger(k), t(k)$ and $r(k)$ form a RT algebra.

Proof. Direct calculation. $\qquad\square$

Note, *en passant*, that if one imposes on $R(k)$ the additional relation $R(k)R(-k) = \mathbb{I}$, one gets $T(k) = 0$ and one recovers the construction [13] of the boundary algebra $\mathcal{B}_S$. Let us also stress that ρ is *not* identity on the algebra generated by $A(k), A^\dagger(k), t(k)$ and $r(k)$. However, we have the following theorem.

Theorem 3.4. *The coset of $\overline{\mathcal{A}}_S$ by the relation $\rho = id$ is a RT algebra.*

Proof. By coset by the relation $\rho = id$, we mean the coset by the two-sided ideal generated by $\text{Im}(\rho - id)$: this coset is obviously an algebra. $A(k)$ and $A^\dagger(k)$ defined in theorem 3.3 can

be taken as a representative of a generating system for the coset which is thus homomorphic to the RT algebra. $\qquad\square$

To conclude this section, let us remark that, in the EZF algebra, one has the identities:

$$A(k) = t(k)A(k) + r(k)A(-k) \tag{3.14}$$

$$A^\dagger(k) = A^\dagger(k)t(k) + A^\dagger(-k)r(-k). \tag{3.15}$$

These identities show that for the RT algebra constructed in theorem 3.3, the reflection–transmission automorphism (as it was introduced in [12]) is indeed the identity, in agreement with theorem 3.4. However, this automorphism (defined only on the RT algebra) must not be confused with the above homomorphism ρ, defined on the whole EZF algebra.

4. Hamiltonians and their symmetries

It is known that one can associate with any ZF algebra a natural hierarchy of Hamiltonians

$$H_{\mathrm{ZF}}^{(n)} = \int_{\mathbb{R}} \mathrm{d}k\, k^n a^\dagger(k) a(k) \qquad n \in \mathbb{Z}_+. \tag{4.1}$$

They obey $\left[H_{\mathrm{ZF}}^{(n)}, H_{\mathrm{ZF}}^{(m)} \right] = 0$ (so that they can indeed be considered as Hamiltonians) and admit as symmetry algebra the quantum group generated by the well-bred operators:

$$\left[H_{\mathrm{ZF}}^{(n)}, L(k) \right] = 0 \qquad \forall n. \tag{4.2}$$

Note that we have the identity

$$H_{\mathrm{ZF}}^{(2n)} = \int_{\mathbb{R}} \mathrm{d}k\, k^{2n} a^\dagger(k) a(k) = \int_{\mathbb{R}} \mathrm{d}k\, k^{2n} \alpha^\dagger(k) \alpha(k) \tag{4.3}$$

which shows that $\rho\big(H_{\mathrm{ZF}}^{(2n)} \big) = H_{\mathrm{ZF}}^{(2n)}$, so that $H_{\mathrm{ZF}}^{(2n)}$ survives the coset of $\overline{\mathcal{A}}_S$ by $\rho = id$.

In a similar way, for any RT algebra, we introduce

$$H_{\mathrm{RT}}^{(n)} = \int_{\mathbb{R}} \mathrm{d}k\, k^n A^\dagger(k) A(k). \tag{4.4}$$

These Hamiltonians can be viewed as the representative of $H_{\mathrm{ZF}}^{(n)}$ in the coset algebra of theorem 3.4. They obey the following relations:

$$\left[H_{\mathrm{RT}}^{(m)}, H_{\mathrm{RT}}^{(n)} \right] = [(-1)^m - (-1)^n] \int_{\mathbb{R}} \mathrm{d}k\, k^{m+n} A^\dagger(k) r(k) A(-k). \tag{4.5}$$

It shows that Hamiltonians of the same parity form a commutative algebra, and thus define a hierarchy corresponding to the integrable systems with impurity, in accordance with equation (4.3). They act on $A(k)$ and $A^\dagger(k)$ as

$$\left[H_{\mathrm{RT}}^{(n)}, A(k) \right] = -k^n ([\mathbb{I} + t(k)]A(k) + (-1)^n r(k)A(-k)) \tag{4.6}$$

$$\left[H_{\mathrm{RT}}^{(n)}, A^\dagger(k) \right] = k^n (A^\dagger(k)[\mathbb{I} + t(k)] + (-1)^n A^\dagger(-k)r(-k)). \tag{4.7}$$

Then, by a direct calculation and *without using the embedding* $\mathcal{C}_S \subset \overline{\mathcal{A}}_S$, one proves:

Proposition 4.1. *The subalgebra* $\mathcal{K}_S$ *is a symmetry algebra of the hierarchy* $H_{\mathrm{RT}}^{(n)}$:

$$\left[H_{\mathrm{RT}}^{(n)}, t(k) \right] = 0 \qquad and \qquad \left[H_{\mathrm{RT}}^{(n)}, r(k) \right] = 0. \tag{4.8}$$

This result provides an universal model-independent description of the symmetry content of the hierarchy $H_{\text{RT}}^{(n)}$. It is quite remarkable that $r(k)$ and $t(k)$ encode both the particle–impurity interactions and the quantum integrals of motion. This property of $\mathcal{K}_S$ has already been applied [19] with success for studying the symmetries of the $gl(N)$-invariant nonlinear Schrödinger equation on the half line.

Finally, using the embedding of the RT algebra into $\overline{\mathcal{A}}_S$, one shows the following.

Proposition 4.2. *When considering the embedding $\mathcal{C}_S \subset \overline{\mathcal{A}}_S$, the subalgebra $\mathcal{K}_S$ becomes a Hopf coideal of the quantum group $\mathcal{U}_S$, i.e. in algebraic terms one has $\mathcal{K}_S \subset \mathcal{U}_S$ and*

$$\Delta(\mathcal{K}_S) \subset \mathcal{U}_S \otimes \mathcal{K}_S \tag{4.9}$$

where Δ is the the coproduct of $\mathcal{U}_S$.

Proof. The construction (3.7) and (3.8) ensures the algebra embedding. Using the coproduct of $\mathcal{U}_S$, one gets

$$\Delta[t_{ab}(k)] = \sum_{x,y} L_{ax}(k) L_{yb}^{-1}(k) \otimes t_{xy}(k) \tag{4.10}$$

$$\Delta[r_{ab}(k)] = \sum_{x,y} L_{ax}(k) L_{yb}^{-1}(-k) \otimes r_{xy}(k). \tag{4.11}$$

$\square$

Let us observe in conclusion that there is a simple relation between the hierarchies with and without impurity, which reads

$$H_{\text{RT}}^{(n)} = H_{\text{ZF}}^{(n)} + \int_{\mathbb{R}} \mathrm{d}k\, k^n a^\dagger(k)(r(k)a(-k) + t(k)a(k)). \tag{4.12}$$

Equation (4.12) generalizes the result of [14] for the boundary algebra $\mathcal{B}_S$.

5. Conclusions

The results of this paper clarify the connection between EZF and RT algebras. We have shown above that a suitable coset algebra, constructed in terms of the EZF algebra, is homomorphic to the RT algebra. This feature provides a precise mathematical meaning of the physical observation [11, 12] that the introduction of an impurity, preserving the integrability of a system, is equivalent to imposing some supplementary constraints on the system. The latter implement the consistency between the scattering in the bulk and the interaction with the impurity.

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